BLOOM | BSO Bloom School Olympiad

GRADE 12

Bloom Mathematics Olympiad
Sample Paper

Maximum Time : 60 Minutes **Maximum Marks : 60**

INSTRUCTIONS

1. There are 50 Multiple Choice Questions in this paper divided into two sections :

   Section A 40 MCQs; 1 Mark each

   Section B 10 MCQs; 2 Marks each

2. Each question has Four Options out of which **ONLY ONE** is correct.
3. All questions are compulsory.
4. There is no negative marking.
5. No electric device capable of storing and displaying visual information such as calculator and mobile is allowed during the course of the exam.

Roll No.

Student's Name

Section-A (1 Mark each)

1. Let $A+2B=\begin{bmatrix}1 & 2 & 0\\ 6 & -3 & 3\\ -5 & 3 & 1\end{bmatrix}$ and $2A-B=\begin{bmatrix}2 & -1 & 5\\ 2 & -1 & 6\\ 0 & 1 & 2\end{bmatrix}$. If $t_r(A)$ denotes the sum of all diagonal elements of the matrix A, then $t_r(A)-t_r(B)$ has value equal to

(a) 1 (b) 2 (c) 0 (d) 3

2. If $A=\begin{bmatrix}1 & 2 & 2\\ 2 & 1 & -2\\ x & 2 & y\end{bmatrix}$ and $A^T A = AA^T = I$, then xy is equal to

(a) –1 (b) 1 (c) 2 (d) –2

3. Let $A=\begin{bmatrix}2 & b & 1\\ b & b^2+1 & b\\ 1 & b & 2\end{bmatrix}$, where $b>0$. Then, the minimum value of $\frac{\det(A)}{b}$ is

(a) $-\sqrt{3}$ (b) $-2\sqrt{3}$ (c) $2\sqrt{3}$ (d) $\sqrt{3}$

4. The element in the first row and third column of the inverse of the matrix $\begin{bmatrix}1 & 2 & -3\\ 0 & 1 & 2\\ 0 & 0 & 1\end{bmatrix}$ is

(a) –2 (b) 0 (c) 1 (d) 7

5. The value of λ and μ for which the system of linear equations

$x+y+z=2, x+2y+3z=5, x+3y+\lambda z=\mu$

has infinitely many solutions are, respectively

(a) 6 and 8 (b) 5 and 7 (c) 5 and 8 (d) 4 and 9

6. The function $f: R \to \left[-\frac{1}{2}, \frac{1}{2}\right]$ defined as $f(x)=\frac{x}{1+x^2}$ is

(a) invertible
(b) injective but not surjective
(c) surjective but not injective
(d) neither injective nor surjective

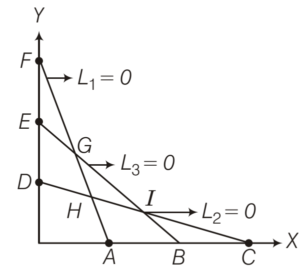
7. Which of the following functions from Z into Z are bijective?

(a) $f(x)=x^3$ (b) $f(x)=x+2$ (c) $f(x)=2x+1$ (d) $f(x)=x^2+1$

8. $f(x)=\frac{\cos x}{\left[\frac{x}{\pi}\right]+\frac{1}{2}}$, (where x is not an integral multiple of π and $[x]$ denote the greatest integer function), is

(a) an odd function
(b) an even function
(c) neither odd nor even
(d) cannot be determined

9. The feasible region for the following constraints $L_1 \leq 0, L_2 \geq 0, L_3 = 0, x \geq 0, y \geq 0$ in the diagram shown is



(a) area *DHF* (b) area *AHC* (c) line segment *EG* (d) line segment *GI*

10. For an LPP, minimise $z = 2x + y$ subject to constraints $5x + 10y \leq 50, x + y \geq 1, y \leq 4$ and $x, y \geq 0$, then z is equal to

(a) 0 (b) 1 (c) 2 (d) 12

11. Let a function $y = y(x)$ be defined parametrically by $x = 2t - |t|, y = t^2 + t|t|$. Then, $y'(x), x > 0$

(a) 0 (b) $4x$ (c) $2x$ (d) does not exist

12. If $y = \tan^{-1}(\sec x - \tan x)$, then $\frac{dy}{dx}$ is equal to

(a) 2 (b) -2 (c) $\frac{1}{2}$ (d) $-\frac{1}{2}$

13. The function $f : R \to R$ defined by $f(x) = e^x$ is

(a) onto (b) one-one
(c) one-one and onto (d) many-one and onto

14. Water is dripping out from a conical funnel of semi-vertical angle $\frac{\pi}{4}$ at the uniform rate of 2 cm^2/s in the surface area, through a tiny hole at the vertex of the bottom. When the slant height of cone is 4 cm, the rate of decrease of the slant height of water, is

(a) $\frac{\sqrt{2}}{4\pi}$ cm/s (b) $\frac{1}{4\pi}$ cm/s (c) $\frac{1}{\pi\sqrt{2}}$ cm/s (d) None of these

15. The point (s) on the curve $y^3 + 3x^2 = 12y$, where the tangent is vertical (parallel to *Y*-axis), is (are)

(a) $\left(\pm\frac{4}{\sqrt{3}}, -2\right)$ (b) $\left(\pm\frac{\sqrt{11}}{3}, 1\right)$ (c) $(0, 0)$ (d) $\left(\pm\frac{4}{\sqrt{3}}, 2\right)$

16. $f(x) = \begin{cases} |x|\cos\left(\frac{1}{x}\right), & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$ is

(a) discontinuous at $x = 0$ (b) continuous at $x = 0$
(c) Does not exist (d) None of the above

17. Let E and F be two independent events such that $P(E) > P(F)$. The probability that both E and F happen is $\frac{1}{12}$ and the probability that neither E nor F happens is $\frac{1}{2}$, then

(a) $P(E) = \frac{1}{3}, P(F) = \frac{1}{4}$ (b) $P(E) = \frac{1}{2}, P(F) = \frac{1}{6}$ (c) $P(E) = 1, P(F) = \frac{1}{12}$ (d) $P(E) = \frac{1}{3}, P(F) = \frac{1}{2}$

18. Let A and B be two events such that the probability of A is $\frac{3}{10}$. Conditional probability of A given B is $\frac{1}{2}$ and the conditional probability of A given complement of B is $\frac{1}{6}$. The probability that exactly one of the events A or B happens

(a) $\frac{3}{10}$ (b) $\frac{7}{10}$ (c) $\frac{1}{10}$ (d) $\frac{9}{10}$

19. A pot contain 5 red and 2 green balls. At random a ball is drawn from this pot. If a drawn ball is green, then put a red ball in the pot and if a drawn ball is red, then put a green ball in the pot, while drawn ball is not replace in the pot. Now, we draw another ball randomly, the probability of second ball to be red is

(a) $\frac{27}{49}$ (b) $\frac{26}{49}$ (c) $\frac{21}{49}$ (d) $\frac{32}{49}$

20. The length of the perpendicular drawn from $(1, 2, 3)$ to the line $\frac{x-6}{3} = \frac{y-7}{2} = \frac{z-7}{-2}$ is

(a) 4 (b) 5 (c) 6 (d) 7

21. The equation of the line passing through the point (3, 0, 1) and parallel to the planes $x + 2y = 0$ and $3y - z = 0$, is

(a) $\frac{x-3}{-2} = \frac{y-0}{1} = \frac{z-1}{3}$ (b) $\frac{x-3}{1} = \frac{y-0}{-2} = \frac{z-1}{3}$

(c) $\frac{x-3}{3} = \frac{y-0}{1} = \frac{z-1}{-2}$ (d) None of these

22. A line with direction cosines proportional to $(2, 7, -5)$ is drawn to intersect the lines $\frac{x-5}{3} = \frac{y-7}{-1} = \frac{z+2}{1}$ and $\frac{x+3}{-3} = \frac{y-3}{2} = \frac{z-6}{4}$, then the coordinate of the points of intersection are

(a) (2, 8, – 3) and (0, 1, 2) (b) (–2, – 8, 3) and (0, –1, 2)

(c) (2, – 8, 3) and (0, 1, – 2) (d) None of these

23. Let $\mathbf{a}$ and $\mathbf{b}$ be two unit vectors and θ is the angle between them. Then, $\mathbf{a} + \mathbf{b}$ is a unit vector, if

(a) $\theta = \frac{\pi}{4}$ (b) $\theta = \frac{\pi}{3}$ (c) $\theta = \frac{\pi}{2}$ (d) $\theta = \frac{2\pi}{3}$

24. A unit vector coplanar with $\hat{\mathbf{i}} + \hat{\mathbf{j}} + 2\hat{\mathbf{k}}$ and $\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + \hat{\mathbf{k}}$ and perpendicular to $\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}$ is

(a) $\left(\frac{\hat{\mathbf{j}} - \hat{\mathbf{k}}}{\sqrt{2}}\right)$ (b) $\left(\frac{\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}}{\sqrt{3}}\right)$ (c) $\left(\frac{\hat{\mathbf{i}} + \hat{\mathbf{j}} + 2\hat{\mathbf{k}}}{\sqrt{6}}\right)$ (d) $\left(\frac{\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + \hat{\mathbf{k}}}{\sqrt{6}}\right)$

25. If $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = -5\mathbf{a} + 4\mathbf{b}$ and $\mathbf{a} \cdot \mathbf{b} = 3$, then $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$ is equal to

(a) $5\mathbf{b} - 3\mathbf{c}$ (b) $3\mathbf{c} - 4\mathbf{b}$ (c) $3\mathbf{b} - 5\mathbf{c}$ (d) $4\mathbf{b} - 3\mathbf{c}$

26. The area bounded by the curve $|x| + y = 1$ and axis of X is

(a) 1 sq unit (b) 2 sq units (c) 8 sq units (d) None of these

27. For the curve $y = 5x - 2x^3$, if x increases at the rate of 2 units/s, then the rate at which the slope of curve is changing when $x = 3$, is

(a) –78 units/s (b) –72 units/s (c) –36 units/s (d) –18 units/s

28. The maximum value of $\frac{\ln x}{x}$ in $(2, \infty)$ is

(a) 1 (b) e (c) $2/e$ (d) $1/e$

29. A continuously differential function $\phi(x)$ in $(0, \pi)$ satisfying $y' = 1 + y^2$, $y(0) = 0 = y(\pi)$, is

(a) $\tan x$ (b) $x(x - \pi)$ (c) $(x - \pi)(1 - e^x)$ (d) Not possible

30. The differential equation that represents all parabolas each of which has a latus rectum $4a$ and whose axes are parallel to X-axis, is

(a) $a\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^3 = 0$ (b) $2a\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^3 = 0$

(c) $2a\frac{d^2y}{dx^2} - \left(\frac{dy}{dx}\right)^3 = 0$ (d) $a\frac{d^2y}{dx^2} - \left(\frac{dy}{dx}\right)^3 = 0$

31. Solution of the differential equation $\frac{dx}{dy} - \frac{x\log_e x}{1+\log_e x} = \frac{e^y}{1+\log_e x}$, if $y(1) = 0$, is

(a) $x^x = e^{ye^y}$ (b) $e^y = x^{e^y}$ (c) $x = ye^y$ (d) $y = e^{x^y}$

32. $\int \frac{xe^x}{(1+x)^2}dx$ is equal to

(a) $\frac{e^x}{1+x} + C$ (b) $\frac{-1}{1+x} + C$ (c) $\frac{-e^x}{(1+x)} + C$ (d) $\frac{e^x}{(1+x)^2} + C$

33. $\int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}}dx$ is equal to

(a) 0 (b) $\frac{\pi}{4}$ (c) $\frac{\pi}{2}$ (d) None of these

34. The value of $\int_0^1 \log\sin\left(\frac{\pi x}{2}\right)dx$ is equal to

(a) log2 (b) –log2 (c) log3 (d) 0

35. Let $f(x) = [n + p\sin x], x \in (0, \pi), n \in Z$, p is a prime number and $[\cdot]$ denotes the greatest integer function. The number of points at which $f(x)$ is not differentiable is

(a) p (b) $p - 1$ (c) $2p + 1$ (d) $2p - 1$

36. If $f(x)=\begin{cases}\dfrac{\sqrt{1+kx}-\sqrt{1-kx}}{x}, & \text{for} \quad -1\le x<0\\ 2x^2+3x-2, & \text{for} \quad 0\le x\le 1\end{cases}$ is continuous at $x=0$, then k is equal to

(a) – 4 (b) – 3 (c) – 2 (d) – 1

37. Let $f(x)=\begin{cases}x^2\cos e^{1/x} & \text{when} \quad x\ne 0\\ 1 & \text{when} \quad x=0\end{cases}$. Then, $f(x)$ is

(a) discontinuous at $x=0$
(b) continuous but not differentiable at $x=0$
(c) differentiable at $x=0$
(d) $\lim\limits_{x\to 0} f(x)$ exist

38. If $\sin^{-1}\left(\dfrac{2a}{1+a^2}\right)+\cos^{-1}\left(\dfrac{1-a^2}{1+a^2}\right)=\tan^{-1}\left(\dfrac{2x}{1-x^2}\right)$, where $a, x\in\,]0,1[$, then the value of x is

(a) 0 (b) $\dfrac{a}{2}$ (c) a (d) $\dfrac{2a}{1-a^2}$

39. The value of the expression $\tan\left(\dfrac{1}{2}\cos^{-1}\dfrac{2}{\sqrt{5}}\right)$ is

(a) $2+\sqrt{5}$ (b) $\sqrt{5}-2$ (c) $\dfrac{\sqrt{5}+2}{2}$ (d) $5+\sqrt{2}$

40. If $\alpha=\sin^{-1}\left(\dfrac{\sqrt{3}}{2}\right)+\sin^{-1}\left(\dfrac{1}{3}\right)$ and $\beta=\cos^{-1}\left(\dfrac{\sqrt{3}}{2}\right)+\cos^{-1}\left(\dfrac{1}{3}\right)$, then

(a) $\alpha>\beta$ (b) $\alpha=\beta$ (c) $\alpha<\beta$ (d) $\alpha+\beta=2\pi$

Section-B (2 Marks each)

41. A and B are two square matrices such that $A^2B=BA$ and if $(AB)^{10}=A^kB^{10}$. Then, k is

(a) 1001 (b) 1023 (c) 1042 (d) None of these

42. Let $f(x)=x^2, x\in R$. For any $A\subseteq R$, define $g(A)=\{x\in R : f(x)\in A\}$. If $S=[0,4]$, then which one of the following statements is not true?

(a) $f(g(S))=S$ (b) $g(f(S))\ne S$ (c) $g(f(S))\ne g(S)$ (d) $f(g(S))\ne f(S)$

43. If $x=2\sin\theta-\sin 2\theta$ and $y=2\cos\theta-\cos 2\theta$,

$\theta\in[0,2\pi]$, then $\dfrac{d^2y}{dx^2}$ at $\theta=\pi$ is

(a) $-\dfrac{3}{4}$ (b) $\dfrac{3}{4}$ (c) $\dfrac{3}{8}$ (d) $\dfrac{3}{2}$

44. Let $f:[0,2]\rightarrow R$ be a twice differentiable function such that $f''(x)>0$, for all $x\in(0,2)$. If $\phi(x)=f(x)+f(2-x)$, then ϕ is

(a) increasing on (0, 1) and decreasing on (1, 2)

(b) decreasing on (0, 2)

(c) decreasing on (0, 1) and increasing on (1, 2)

(d) increasing on (0, 2)

45. A box has four dice in it. Three of them are fair dice but the fourth one has the number five on all of its faces. A die is chosen at random from the box and is rolled three times and shows up the face five on all the three occassions. The chance that the die chosen was a rigged die, is

(a) $\frac{216}{217}$ (b) $\frac{215}{219}$ (c) $\frac{216}{219}$ (d) None of these

46. Let P be a plane passing through the points (2, 1, 0), (4, 1, 1) and (5, 0, 1) and R be any point (2, 1, 6). Then the image of R in the plane P is

(a) (6, 5, 2) (b) (4, 3, 2) (c) (6, 5, –2) (d) (3, 4, –2)

47. The maximum value of $x^{1/x}$ is

(a) $\frac{1}{e^e}$ (b) e (c) $\frac{1}{e}$ (d) $e^{1/e}$

48. A curve $y=f(x)$ passes through the point $P(1,1)$. The normal to the curve at point P is $a(y-1)+(x-1)=0$. If the slope of the tangent at any point on the curve is proportional to the ordinate at that point, then the equation of the curve is

(a) $y=e^{ax}-1$ (b) $y=e^{ax}+1$ (c) $y=e^{ax}+a$ (d) $y=e^{a(x-1)}$

49. $\int\frac{\sin^{-1}x}{(1-x^2)^{3/2}}dx$ is equal to

(a) $(\sin^{-1}x)\frac{x}{\sqrt{1-x^2}}+\log(\sqrt{1-x^2})+C$ (b) $\frac{\sin^{-1}x}{\sqrt{1-x^2}}+\log(\sqrt{1-x^2})+C$

(c) $\frac{x\sin^{-1}x}{\sqrt{1-x^2}}-\log(\sqrt{1-x^2})+C$ (d) None of these

50. If $a_1,a_2,a_3,\ldots,a_n$ are in AP with common difference 5 and if $a_ia_j\neq-1$ for $i,j=1,2,\ldots,n$,

then $\tan^{-1}\left(\frac{5}{1+a_1a_2}\right)+\tan^{-1}\left(\frac{5}{1+a_2a_3}\right)+\cdots+\tan^{-1}\left(\frac{5}{1+a_{n-1}a_n}\right)$ is equal to

(a) $\tan^{-1}\left(\frac{5}{1+a_na_{n-1}}\right)$ (b) $\tan^{-1}\left(\frac{5a_1}{1+a_na_1}\right)$

(c) $\tan^{-1}\left(\frac{5n-5}{1+a_na_1}\right)$ (d) $\tan^{-1}\left(\frac{5n-5}{1+a_1a_{n+1}}\right)$

OMR SHEET

1	a	●	c	d	***2***	a	b	●	d	***3***	a	b	●	d	***4***	a	b	c	●
5	a	b	●	d	***6***	a	b	●	d	***7***	a	●	c	d	***8***	●	b	c	d
9	a	b	●	d	***10***	a	●	c	d	***11***	a	●	c	d	***12***	a	b	c	●
13	●	b	c	d	***14***	●	b	c	d	***15***	a	b	c	●	***16***	a	●	c	d
17	●	b	c	d	***18***	●	b	c	d	***19***	a	b	c	●	***20***	a	b	c	●
21	●	b	c	d	***22***	●	b	c	d	***23***	a	b	c	●	***24***	●	b	c	d
25	a	b	c	●	***26***	●	b	c	d	***27***	a	●	c	d	***28***	a	b	c	●
29	a	b	c	●	***30***	a	●	c	d	***31***	●	b	c	d	***32***	●	b	c	d
33	a	●	c	d	***34***	a	●	c	d	***35***	a	b	c	●	***36***	a	b	●	d
37	●	b	c	d	***38***	a	b	c	●	***39***	a	●	c	d	***40***	a	b	●	d
41	a	●	c	d	***42***	a	b	●	d	***43***	a	b	●	d	***44***	a	b	●	d
45	a	b	●	d	***46***	a	b	●	d	***47***	●	b	c	d	***48***	a	b	c	●
49	●	b	c	d	***50***	a	b	●	d										

Answers with Hints

1. $A + 2B = \begin{bmatrix} 1 & 2 & 0 \\ 6 & -3 & 3 \\ -5 & 3 & 1 \end{bmatrix}$ ...(i)

$$2A - B = \begin{bmatrix} 2 & -1 & 5 \\ 2 & -1 & 6 \\ 0 & 1 & 2 \end{bmatrix}$$

$$4A - 2B = \begin{bmatrix} 4 & -2 & 10 \\ 4 & -2 & 12 \\ 0 & 2 & 4 \end{bmatrix} \quad \text{...(ii)}$$

On adding Eqs. (i) and (ii), we get

$$5A = \begin{bmatrix} 5 & 0 & 10 \\ 10 & -5 & 15 \\ -5 & 5 & 5 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 2 & -1 & 3 \\ -1 & 1 & 1 \end{bmatrix} \text{and } B = \begin{bmatrix} 0 & 1 & -1 \\ 2 & -1 & 0 \\ -2 & 1 & 0 \end{bmatrix}$$

$\Rightarrow \quad t_r(A) = 1 - 1 + 1 = 1$

$\Rightarrow \quad t_r(B) = 0 - 1 + 0 = -1$

$\therefore \quad t_r(A) - t_r(B) = 2$

2. Since, A is orthogonal, each row is orthogonal to the other rows.

$\Rightarrow \quad R_1 \cdot R_3 = 0$

$\Rightarrow \quad x + 4 + 2y = 0$ ...(i)

Also, $\quad R_2 \cdot R_3 = 0$

$\Rightarrow \quad 2x + 2 - 2y = 0$ ...(ii)

On solving Eqs. (i) and (ii), we get

$x = -2, y = -1$

$\therefore \quad xy = 2$

3. Given, matrix $A = \begin{bmatrix} 2 & b & 1 \\ b & b^2+1 & b \\ 1 & b & 2 \end{bmatrix}, b > 0$

So, $\det(A) = |A| = \begin{vmatrix} 2 & b & 1 \\ b & b^2+1 & b \\ 1 & b & 2 \end{vmatrix}$

$= 2[2(b^2+1) - b^2] - b(2b - b) + 1(b^2 - b^2 - 1)$

$= 2[2b^2 + 2 - b^2] - b^2 - 1$

$= 2b^2 + 4 - b^2 - 1 = b^2 + 3$

$$\Rightarrow \qquad \frac{\det(A)}{b} = \frac{b^2+3}{b} = b + \frac{3}{b}$$

Now, by $\text{AM} \geq \text{GM}$, we get

$$\frac{b+\frac{3}{b}}{2} \geq \left(b \times \frac{3}{b}\right)^{1/2} \qquad [\because b > 0]$$

$$\Rightarrow \qquad b + \frac{3}{b} \geq 2\sqrt{3}$$

So, minimum value of $\frac{\det(A)}{b} = 2\sqrt{3}$.

4. Let $\quad A = \begin{bmatrix} 1 & 2 & -3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$

$$\therefore \qquad |A| = \begin{bmatrix} 1 & 2 & -3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} = \begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix} = 1$$

and $\text{adj}\, A = \begin{bmatrix} 1 & -2 & 7 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$

Hence, $A^{-1} = \frac{1}{|A|}\,\text{adj}\, A = \begin{bmatrix} 1 & -2 & 7 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$

So, required element $= A^{-1}_{13} = 7$

5. The system of linear equations

$$x + y + z = 2$$
$$x + 2y + 3z = 5$$
$$x + 3y + \lambda z = \mu$$

has infinitely many solutions, so

$$\Delta = 0 \Rightarrow \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & \lambda \end{vmatrix} = 0$$

$$\Rightarrow \qquad 1(2\lambda - 9) - 1(\lambda - 3) + 1(3 - 2) = 0 \Rightarrow \lambda = 5$$

and $\qquad \Delta_3 = 0 \Rightarrow \begin{vmatrix} 1 & 1 & 2 \\ 1 & 2 & 5 \\ 1 & 3 & \mu \end{vmatrix} = 0$

$$\Rightarrow \qquad 1(2\mu - 15) - 1(\mu - 5) + 2(3 - 2) = 0 \Rightarrow \mu = 8$$

6. We have, $f(x) = \frac{x}{1+x^2}$

$$\therefore \qquad f\left(\frac{1}{x}\right) = \frac{\frac{1}{x}}{1+\frac{1}{x^2}} = \frac{x}{1+x^2} = f(x)$$

$\therefore \quad f\left(\frac{1}{2}\right) = f(2)$ or $f\left(\frac{1}{3}\right) = f(3)$ and so on.

So, $f(x)$ is many-one function.

Again, let $y = f(x) \Rightarrow y = \frac{x}{1+x^2}$

$\Rightarrow \quad y + x^2 y = x \Rightarrow yx^2 - x + y = 0$

As, $\quad x \in R$

$\therefore \quad (-1)^2 - 4(y)(y) \geq 0 \Rightarrow 1 - 4y^2 \geq 0 \Rightarrow y \in \left[\frac{-1}{2}, \frac{1}{2}\right]$

$\therefore$ Range = Codomain = $\left[\frac{-1}{2}, \frac{1}{2}\right]$

So, $f(x)$ is surjective.

Hence, $f(x)$ is surjective but not injective.

7. The function $f(x) = x + 2$ is one-one as for $x_1, x_2 \in Z$. Consider, $f(x_1) = f(x_2)$

$$\Rightarrow \quad x_1 + 2 = x_2 + 2$$
$$\Rightarrow \quad x_1 = x_2$$

Also, let $y \in$ codomain of $f = Z$ such that

$$y = f(x)$$
$$\Rightarrow \quad y = x + 2$$
$$\Rightarrow \quad x = y - 2 \in Z \text{ for all } y \in Z$$

$\therefore f$ is onto.

Hence, $f(x) = x + 2$ is bijective.

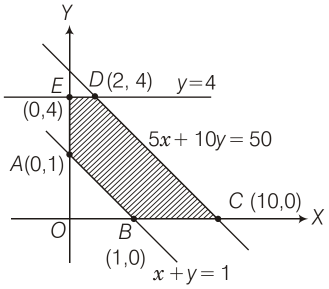
8. $f(-x) = \dfrac{\cos(-x)}{\left[-\frac{x}{\pi}\right] + \frac{1}{2}} = \dfrac{\cos x}{-\left[\frac{x}{\pi}\right] - 1 + \frac{1}{2}}$ $\quad \left(\text{as } x \neq n\pi \Rightarrow \frac{x}{\pi} \notin I, \text{ so as } \left[-\frac{x}{\pi}\right] = -\left[\frac{x}{\pi}\right] - 1\right)$

$$= \frac{-\cos x}{\left[\frac{x}{\pi}\right] + \frac{1}{2}} = -f(x) \Rightarrow f(x) \text{ is an odd function.}$$

Hence, (a) is correct answer.

9. In the given figure, the feasible region for given constraints is the line segment EG.

10. Feasible region is $ABCDEA$ and vertices of the feasible region are $A(0, 1)$, $B(1, 0)$, $C(10, 0)$, $D(2, 4)$ and $E(0, 4)$.



Thus, minimum value of objective function is 1 at (0, 1).

$\therefore \qquad z = 2\times 0+1=1$

11. $\because\ x = 2t - |t| = \begin{cases} t, & t\ge 0 \\ 3t, & t<0 \end{cases}$,

$\therefore \qquad t = \begin{cases} x, & x\ge 0 \\ \dfrac{x}{3}, & x<0 \end{cases}$

$\therefore \quad y = t^2 + t|t| = \begin{cases} 2t^2, & t\ge 0 \\ 0, & x<0 \end{cases}$

$= \begin{cases} 2x^2, & x\ge 0 \\ 0, & x<0 \end{cases}$

Hence, $y'(x) = \begin{cases} 4x, & x>0 \\ 0, & x<0 \end{cases}$

$\therefore$ We can not find $\dfrac{dx}{dt}$ as the derivative does not exist at $t=0$.

12. We have, $y = \tan^{-1}(\sec x - \tan x)$

$\Rightarrow \qquad \dfrac{dy}{dx} = \dfrac{d}{dx}\tan^{-1}\left(\dfrac{1-\sin x}{\cos x}\right)$

$\Rightarrow \qquad \dfrac{dy}{dx} = \dfrac{d}{dx}\tan^{-1}\left(\dfrac{\cos\frac{x}{2}-\sin\frac{x}{2}}{\cos\frac{x}{2}+\sin\frac{x}{2}}\right)$

$= \dfrac{d}{dx}\tan^{-1}\left[\tan\left(\dfrac{\pi}{4}-\dfrac{x}{2}\right)\right]$

$= \dfrac{d}{dx}\left(\dfrac{\pi}{4}-\dfrac{x}{2}\right) = -\dfrac{1}{2}$

13. Let $f: R\to R$ be the function defined by

$$f(x) = e^x$$

Now, let $x_1, x_2 \in R$ such that

$$f(x_1) = f(x_2)$$

$\Rightarrow \qquad e^{x_1} = e^{x_2}$

$\Rightarrow \qquad x_1 = x_2$

$\Rightarrow f$ is one-one.

Now, let $y\in R$ such that

$$f(x) = y$$

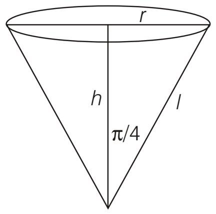
$\Rightarrow \qquad e^x = y$

$\Rightarrow \qquad x = \log y$ (taking log on both sides)

For zero and negative real number in the codomain of f, there does not exist any pre image in the domain of f.

$\therefore$ The function f is not onto.

14. If S represents the surface area, then



$$\frac{dS}{dt} = 2\ \text{cm}^2/\text{s}$$

$$S = \pi r l = \pi l \cdot \sin\frac{\pi}{4} l = \frac{\pi}{\sqrt{2}} l^2$$

Therefore, $\quad \dfrac{dS}{dt} = \dfrac{2\pi}{\sqrt{2}} l \cdot \dfrac{dl}{dt} = \sqrt{2}\pi l \cdot \dfrac{dl}{dt}$

when $l = 4$ cm, $\dfrac{dl}{dt} = \dfrac{l}{\sqrt{2}\pi \cdot 4} \cdot 2 = \dfrac{1}{2\sqrt{2}\pi} = \dfrac{\sqrt{2}}{4\pi}$ cm/s

15. Given, curve is $y^3 + 3x^2 = 12y$

On differentiating w.r.t. y, we get

$$3y^2 \frac{dy}{dx} + 6x = 12\frac{dy}{dx}$$

$\Rightarrow \quad \dfrac{dy}{dx}(3y^2 - 12) + 6x = 0$

$\Rightarrow \quad \dfrac{dy}{dx} = \dfrac{6x}{12 - 3y^2}$

$\Rightarrow \quad \dfrac{dx}{dy} = \dfrac{12 - 3y^2}{6x}$

Since, tangent is parallel to Y-axis.

$$\frac{dx}{dy} = 0 \Rightarrow 12 - 3y^2 = 0$$

$\Rightarrow \quad y^2 = 4 \Rightarrow y = \pm 2$

Then, at $y = 2$, $x = \pm\dfrac{4}{\sqrt{3}}$

At $y = -2$, x cannot be real.

$\therefore$ The required point is $\left(\pm\dfrac{4}{\sqrt{3}}, 2\right)$.

16. To check the continuity at $x = 0$

$$\text{LHL} = \lim_{x\to 0^-} f(x) = \lim_{x\to 0} f(0 - h) = \lim_{h\to 0} |-h| \cos\left(\frac{1}{-h}\right)$$

$$= \lim_{h\to 0} h\cos\left(\frac{1}{h}\right) \qquad (\because -1 \le \cos x \le 1 \ \forall\ x \in R)$$

$$= 0 \text{ (an oscillating value between } -1 \text{ and } 1) = 0$$

$$\text{RHL} = \lim_{x\to 0^+} f(x) = \lim_{h\to 0} (0 + h) = \lim_{h\to 0} (h) \lim_{h\to 0} h\cos\frac{1}{h}$$

$= 0$ (an oscillating number between -1 and 1) $= 0$ and $f(0) = 0$

Thus, LHL = RHL = $f(0) = 0$. Hence, function is continuous at $x = 0$.

17. $P(E \cap F) = P(E)\,P(F) = \frac{1}{12}$ ...(i)

$P(E^c \cap F^c) = P(E^c) \cdot P(F^c) = \frac{1}{2}$

$\Rightarrow \quad (1 - P(E))(1 - P(F)) = \frac{1}{2}$ ...(ii)

On solving Eqs. (i) and (ii), we get

$$P(E) = \frac{1}{3} \text{ and } P(F) = \frac{1}{4}, \text{ as } P(E) > P(F)$$

18. Given, $P(A) = \frac{3}{10}$; $P\left(\frac{A}{B}\right) = \frac{1}{2}$; $P\left(\frac{A}{B'}\right) = \frac{1}{6}$

Probability of happening of exactly one of the events A or $B = P(A) + P(B) - 2P(A \cap B)$ ...(i)

Also, $P\left(\frac{A}{B}\right) = \frac{P(A \cap B)}{P(B)} = \frac{1}{2} \Rightarrow 2P(A \cap B) = P(B)$

From Eq. (i), required probability $= \frac{3}{10} + P(B) - P(B) = \frac{3}{10}$

19. $P(G1)$ = Probability that Ist ball drawn is green $= \frac{2}{7}$

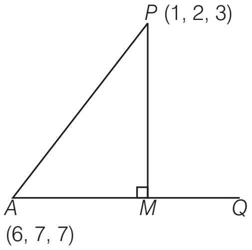
$P(R1)$ = Probability that Ist ball drawn in red $= \frac{5}{7}$

$P(R)$ = Probability that 2nd ball drawn is red.

$= P(G1) \cdot \frac{6}{7} + P(R1) \cdot \frac{4}{7}$

$= \frac{2}{7} \times \frac{6}{7} + \frac{5}{7} \times \frac{4}{7} = \frac{32}{49}$

20. Direction cosines of given line are $\frac{3}{\sqrt{17}}, \frac{2}{\sqrt{17}}, \frac{-2}{\sqrt{17}}$



$\therefore \quad AM = \left|(6-1) \cdot \frac{3}{\sqrt{17}} + (7-2) \cdot \frac{2}{\sqrt{17}} + (7-3) \cdot \frac{-2}{\sqrt{17}}\right|$

$= \sqrt{17}$

$AP = \sqrt{(6-1)^2 + (7-2)^2 + (7-3)^2}$

$= \sqrt{25 + 25 + 16} = \sqrt{66}$

$\therefore$ Length of perpendicular $PM = \sqrt{AP^2 - AM^2}$

$= \sqrt{66 - 17} = \sqrt{49} = 7$

21. Let a, b and c be the direction ratios of the required line. Then, its equation is

$$\frac{x-3}{a}=\frac{y-0}{b}=\frac{z-1}{c} \quad \text{...(i)}$$

Since, Eq. (i) is parallel to the planes $x+2y+0z=0$ and $0x+3y-z=0$. Therefore, normal to the plane is perpendicular to the line.

$\therefore \quad a(1)+b(2)+c(0)=0$ and $a(0)+b(3)+c(-1)=0$

On solving these two equations by cross-multiplication, we obtain

$$\frac{a}{(2)(-1)-(0)(3)}=\frac{b}{(0)(0)-(1)(-1)}=\frac{c}{(1)(3)-(0)(2)}$$

$$\Rightarrow \quad \frac{a}{-2}=\frac{b}{1}=\frac{c}{3}=\lambda \text{ (say)}$$

$$\Rightarrow \quad a=-2\lambda,\ b=\lambda,\ c=3\lambda$$

On substituting the values of a, b and c in Eq. (i) , we obtain the equation of the required line as

$$\frac{x-3}{-2}=\frac{y-0}{1}=\frac{z-1}{3}$$

Alternate Method

The required line passes through the point having its position vector $\mathbf{a}=3\hat{\mathbf{i}}+\hat{\mathbf{k}}$ and is parallel to the planes $x+2y=0$ and $3y-z=0$. So, it is perpendicular to their normals $\mathbf{n}_1=\hat{\mathbf{i}}+2\hat{\mathbf{j}}$ and $\mathbf{n}_2=3\hat{\mathbf{j}}-\hat{\mathbf{k}}$, respectively.

Consequently, the required line is parallel to the vector.

$$\mathbf{b}=\mathbf{n}_1\times\mathbf{n}_2=\begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ 1 & 2 & 0\\ 0 & 3 & -1\end{vmatrix}=-2\hat{\mathbf{i}}+\hat{\mathbf{j}}+3\hat{\mathbf{k}}$$

Hence, the equation of the required line is

$$\mathbf{r}=\mathbf{a}+\lambda\mathbf{b} \text{ or } \mathbf{r}=(3\hat{\mathbf{i}}+\hat{\mathbf{k}})+\lambda(-2\hat{\mathbf{i}}+\hat{\mathbf{j}}+3\hat{\mathbf{k}})$$

22. The given equations

$$\frac{x-5}{3}=\frac{y-7}{-1}=\frac{z+2}{1} \quad \text{...(i)}$$

$$\frac{x+3}{-3}=\frac{y-3}{2}=\frac{z-6}{4} \quad \text{...(ii)}$$

Any point P on Eq.(i) is $(3r_1+5, -r_1+7, r_1-2)$ and any point Q on Eq. (ii) is $(-3r_2-3, 2r_2+3, 4r_2+6)$ the direction ratios of PQ are

$$(3r_1+3r_2+8, -r_1-2r_2+4, r_1-4r_2-8) \quad \text{...(iii)}$$

Suppose the line with DR's 2, 7, –5 will be proportional to the DR's given by Eq.(iii)

$$\therefore \quad \frac{(3r_1+3r_2+8)}{2}=\frac{-r_1-2r_2+4}{7}=\frac{r_1-4r_2-8}{-5} \quad \text{...(iv)}$$

On solving Eq.(iv), we get $r_1=r_2=-1$

So, point of intersection are $P(2, 8, -3)$ and $Q(0, 1, 2)$

23. Let a and b be two unit vectors and θ be the angle between them.

Then, $|\mathbf{a}| = |\mathbf{b}| = 1$

Now, $(\mathbf{a} + \mathbf{b})$ is a unit vector, if

$$|\mathbf{a} + \mathbf{b}| = 1 \Rightarrow (\mathbf{a} + \mathbf{b})^2 = 1$$

$$\Rightarrow (\mathbf{a} + \mathbf{b}) \cdot (\mathbf{a} + \mathbf{b}) = 1 \Rightarrow \mathbf{a} \cdot \mathbf{a} + \mathbf{a} \cdot \mathbf{b} + \mathbf{b} \cdot \mathbf{a} + \mathbf{b} \cdot \mathbf{b} = 1$$

$$\Rightarrow |\mathbf{a}|^2 + |\mathbf{b}|^2 + 2\mathbf{a} \cdot \mathbf{b} = 1 \quad (\because \mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a})$$

$$\Rightarrow 1^2 + 1^2 + 2\mathbf{a} \cdot \mathbf{b} = 1$$

$$\Rightarrow \mathbf{a} \cdot \mathbf{b} = -\frac{1}{2} \Rightarrow |\mathbf{a}||\mathbf{b}|\cos\theta = -\frac{1}{2}$$

$$\Rightarrow 1 \times 1 \times \cos\theta = -\frac{1}{2}$$

$$\Rightarrow \cos\theta = -\frac{1}{2} \Rightarrow \theta = \frac{2\pi}{3}$$

24. Let unit vector is $a\hat{\mathbf{i}} + b\hat{\mathbf{j}} + c\hat{\mathbf{k}}$

$\because a\hat{\mathbf{i}} + b\hat{\mathbf{j}} + c\,\hat{\mathbf{k}}$ is perpendicular to $\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}$

Then, $a + b + c = 0$ ...(i)

and $a\hat{\mathbf{i}} + b\hat{\mathbf{j}} + c\hat{\mathbf{k}}$, $(\hat{\mathbf{i}} + \hat{\mathbf{j}} + 2\hat{\mathbf{k}})$ and $(\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + \hat{\mathbf{k}})$ are coplanar.

$$\therefore \begin{vmatrix} a & b & c \\ 1 & 1 & 2 \\ 1 & 2 & 1 \end{vmatrix} = 0$$

$$\Rightarrow -3a + b + c = 0 \qquad \text{...(ii)}$$

From Eqs. (i) and (ii), we get

$$a = 0 \text{ and } c = -b$$

$\because a\hat{\mathbf{i}} + b\hat{\mathbf{j}} + c\,\hat{\mathbf{k}}$ is a unit vector, then

$$a^2 + b^2 + c^2 = 1$$

$$\Rightarrow 0 + b^2 + b^2 = 1$$

$$\Rightarrow b = \frac{1}{\sqrt{2}}$$

$$\therefore a\hat{\mathbf{i}} + b\hat{\mathbf{j}} + c\hat{\mathbf{k}} = \frac{1}{\sqrt{2}}\hat{\mathbf{j}} - \frac{1}{\sqrt{2}}\hat{\mathbf{k}} = \frac{\hat{\mathbf{j}} - \hat{\mathbf{k}}}{\sqrt{2}}$$

25. $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = (\mathbf{c} \cdot \mathbf{a})\mathbf{b} - (\mathbf{c} \cdot \mathbf{b})\mathbf{a} = -5\mathbf{a} + 4\mathbf{b}$

$$\therefore \mathbf{c} \cdot \mathbf{a} = 4 \text{ and } \mathbf{c} \cdot \mathbf{b} = 5$$

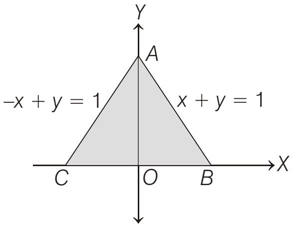
$$\Rightarrow \mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$$

$$\Rightarrow 4\mathbf{b} - 3\mathbf{c}$$

26. Given, curve is $|x| + y = 1$

$\therefore$ Curve is $x + y = 1$, when $x \geq 0$ and $-x + y = 1$, when $x < 0$

The graph of the curve is as given in the figure.



$\therefore$ Required area = Area $CAOC$ + Area $OABO$

$$= \int_{-1}^{0} y\,dx + \int_{0}^{1} y\,dx$$

$$= \int_{-1}^{0} (x+1)\,dx + \int_{0}^{1} (1-x)\,dx$$

$$= \left[\frac{x^2}{2} + x\right]_{-1}^{0} + \left[x - \frac{x^2}{2}\right]_{0}^{1}$$

$$= \left[0 - \left(\frac{1}{2} - 1\right)\right] + \left[\left(1 - \frac{1}{2}\right) - 0\right]$$

$$= \frac{1}{2} + \frac{1}{2} = 1 \text{ sq unit}$$

27. Slope of curve $= \dfrac{dy}{dx} = 5 - 6x^2$

$$\Rightarrow \quad \frac{d}{dt}\left(\frac{dy}{dx}\right) = -12x \cdot \frac{dx}{dt}$$

$$= -12 \cdot (3) \cdot (2) = -72 \text{ units/s}$$

Thus, slope of curve is decreasing at the rate of 72 units/s when x is increasing at the rate of 2 units/s.

28. Let $y = \dfrac{\ln x}{x}$

$$\frac{dy}{dx} = \frac{x.\frac{1}{x} - \ln x.1}{x^2} = \frac{1 - \log x}{x^2}$$

For maxima, put $\dfrac{dy}{dx} = 0$

$$\Rightarrow \quad \frac{1 - \ln x}{x^2} = 0 \Rightarrow x = e$$

Now, $\dfrac{d^2y}{dx^2} = \dfrac{x^2\left(-\frac{1}{x}\right) - (1 - \ln x)\,2x}{(x^2)^2}$

At $\quad x = e, \dfrac{d^2y}{dx^2} < 0,$

$\therefore$ The maximum value at $x = e$ is $y = \dfrac{1}{e}$

29. Given that, $\frac{dy}{dx} = 1 + y^2 \Rightarrow \frac{dy}{1+y^2} = dx$

On integrating both sides, we get

$$\int \frac{dy}{1+y^2} = \int dx$$

$\Rightarrow$ $\tan^{-1} y = x + C$

At $x = 0$, $y = 0$, then $C = 0$

At $x = \pi$, $y = 0$, then $\tan^{-1} 0 = \pi + C \Rightarrow C = -\pi$

$\therefore \tan^{-1} y = x \Rightarrow y = \tan x = \phi(x)$

Therefore, solution becomes $y = \tan x$.

But $\tan x$ is not continuous function in $(0, \pi)$.

So, $\phi(x)$ is not possible in $(0, \pi)$.

30. Equation of the family of such parabola is

$$(y - k)^2 = 4a(x - h) \quad \text{...(i)}$$

where h and k are arbitrary constants.

On differentiating w.r.t, x we get

$$(y - k)\frac{dy}{dx} = 2a \quad \text{...(ii)}$$

On differentiating again,

$$(y - k)\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = 0 \quad \text{...(iii)}$$

On putting value of $(y - k)$ from Eq. (ii) in Eq. (iii), we get

$$2a\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^3 = 0,$$

which is the required differential equation.

31. $(1 + \log_e x)\frac{dx}{dy} - x\log_e x = e^y$

On putting $x\log_e x = t \Rightarrow (1 + \log_e x)dx = dt$

$\therefore$ $\frac{dt}{dy} - t = e^y \Rightarrow te^{-y} = \int e^{-y}e^y dy + C$

$\Rightarrow$ $t = Ce^y + ye^y$

$\Rightarrow$ $x\log_e x = (C + y)e^y$

Since, $y(1) = 0$ i.e. $C = 0$

$\therefore$ $ye^y = x\log_e x$

$\Rightarrow$ $x^x = e^{ye^y}$

32. Let $I=\int \frac{xe^x}{(1+x)^2}dx=\int e^x\frac{(x+1)-1}{(1+x)^2}dx$

$$=\int e^x\left[\frac{1}{(1+x)}-\frac{1}{(1+x)^2}\right]dx$$

Let $f(x)=\frac{1}{1+x}\Rightarrow f'(x)=-\frac{1}{(1+x)^2}$

We know that $\int e^x\{f(x)+f'(x)\}\,dx=e^x\,f(x)$

$$\Rightarrow \quad I=\int e^x\left\{\frac{1}{1+x}-\frac{1}{(1+x)^2}\right\}dx=\frac{e^x}{1+x}+C$$

33. Firstly, reduce the integrand into simplest form by using the property $\int_0^a f(x)dx=\int_0^a f(a-x)dx$, add them and integrate.

Let $I=\int_0^{\pi/2}\frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}dx$ ...(i)

Then, $I=\int_0^{\pi/2}\frac{\sqrt{\sin\left(\frac{\pi}{2}-x\right)}}{\sqrt{\sin\left(\frac{\pi}{2}-x\right)}+\sqrt{\cos\left(\frac{\pi}{2}-x\right)}}dx$ $\left[\because \int_0^a f(x)dx=\int_0^a f(a-x)dx\right]$

$\Rightarrow \quad I=\int_0^{\pi/2}\frac{\sqrt{\cos x}}{\sqrt{\cos x}+\sqrt{\sin x}}dx$ ...(ii)

$\left[\because \sin\left(\frac{\pi}{2}-x\right)=\cos x \text{ and } \cos\left(\frac{\pi}{2}-x\right)=\sin x\right]$

On adding Eqs. (i) and (ii), we get

$$2I=\int_0^{\pi/2}\frac{\sqrt{\sin x}+\sqrt{\cos x}}{\sqrt{\sin x}+\sqrt{\cos x}}dx$$

$$=\int_0^{\pi/2}1\,dx=[x]_0^{\pi/2}=\frac{\pi}{2}-0$$

$$\Rightarrow \quad I=\frac{\pi}{4}$$

34. Let $I=\int_0^1\log\sin\left(\frac{\pi x}{2}\right)dx$

On putting $\frac{\pi x}{2}=t\Rightarrow dx=\frac{2}{\pi}dt$

$\therefore \quad I=\frac{2}{\pi}\int_0^{\pi/2}\log\sin t\,dt=\frac{2}{\pi}I_1,$ ...(i)

where $I_1=\int_0^{\pi/2}\log\sin t\,dt=\int_0^{\pi/2}\log\sin\left(\frac{\pi}{2}-t\right)dt$

$$=\int_0^{\pi/2}\log\cos t\,dt$$

$\therefore \quad 2I_1=\int_0^{\pi/2}(\log\sin t+\log\cos t)\,dt$

$$=\int_0^{\pi/2}\log(\sin t\cos t)\,dt=\int_0^{\pi/2}\log\left(\frac{\sin 2t}{2}\right)dt$$

$$= \int_0^{\pi/2} (\log \sin 2t - \log 2)\, dt$$

$$= \frac{1}{2}\int_0^{\pi} \log \sin z\, dz - \log 2\int_0^{\pi/2} dt$$ [On putting $2t = z$ in first integral, $\therefore 2dx = dz$]

$$= \frac{1}{2}\cdot 2\int_0^{\pi/2} \log \sin z\, dz - \frac{\pi}{2}\log 2$$

$$\Rightarrow \quad 2I_1 = I_1 - \frac{\pi}{2}\log 2 \Rightarrow I_1 = -\frac{\pi}{2}\log 2$$

$\therefore$ From Eq. (i), $I = \frac{2}{\pi}\left(-\frac{\pi}{2}\log 2\right) = -\log 2$

35. $f(x) = [n + p\sin x] = n + [p\sin x]$

$f(x)$ is not differentiable at those points, where $p\sin x$ is an integer.

$p\sin x$ is an integer if $\sin x = 1, -1$ and $\frac{x}{p}$

i.e. $x = \frac{\pi}{2}, \frac{-\pi}{2}, \sin^{-1}\frac{r}{p}, \pi - \sin^{-1}\frac{r}{p}$, where $0 \le r \le p-1$

But $x = \frac{-\pi}{2}, 0$ can not be positive

$\therefore$ Function is not differentiable at $x = \frac{\pi}{2}$,

$\sin^{-1}\frac{r}{p}, \pi - \sin^{-1}\frac{r}{p}$, where $0 < r < p-1$

So, the required number of points are

$$1 + 2(p-1) = 2p - 1$$

36. LHL $= \lim_{x \to 0^-} \frac{\sqrt{1+kx} - \sqrt{1-kx}}{x}$

$$= \lim_{x \to 0^-} \frac{2kx}{x(\sqrt{1+kx} + \sqrt{1-kx})} = k$$

RHL $= \lim_{x \to 0^+} (2x^2 + 3x - 2) = -2$

Since, it is given that $f(x)$ is continuous.

$\therefore$ LHL = RHL $\Rightarrow$ $k = -2$

37. $f(x) = \begin{cases} x^2 \cos e^{1/x}, & x \ne 0 \\ 1, & x = 0 \end{cases}$

LHL $= \lim_{x \to 0^-} f(x) = 0$

RHL $= \lim_{x \to 0^+} f(x) = 0$

and $\lim_{x \to 0} f(x) = 0$

$\Rightarrow \lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) \ne f(0)$

$\Rightarrow f(x)$ is discontinuous function at $x = 0$.

38. Given, $\sin^{-1}\left(\frac{2a}{1+a^2}\right)+\cos^{-1}\left(\frac{1-a^2}{1+a^2}\right)=\tan^{-1}\left(\frac{2x}{1-x^2}\right)$

$$\therefore \quad 2\tan^{-1}a + 2\tan^{-1}a = 2\tan^{-1}x$$

$$\Rightarrow \quad 4\tan^{-1}a = 2\tan^{-1}x$$

$$\Rightarrow \quad 2\tan^{-1}a = \tan^{-1}x$$

$$\Rightarrow \quad \tan^{-1}\frac{2a}{1-a^2} = \tan^{-1}x$$

$$\Rightarrow \quad x = \frac{2a}{1-a^2}$$

39. $\tan\left(\frac{1}{2}\cos^{-1}\frac{2}{\sqrt{5}}\right)=\sqrt{\dfrac{1-\cos\left[\cos^{-1}\left(\frac{2}{\sqrt{5}}\right)\right]}{1+\cos\left[\cos^{-1}\left(\frac{2}{\sqrt{5}}\right)\right]}}$

$$=\sqrt{\frac{1-\frac{2}{\sqrt{5}}}{1+\frac{2}{\sqrt{5}}}}=\sqrt{\frac{\sqrt{5}-2}{\sqrt{5}+2}\times\frac{\sqrt{5}-2}{\sqrt{5}-2}}$$

$$=\sqrt{\frac{(\sqrt{5}-2)^2}{5-4}}=\sqrt{5}-2$$

40. $\alpha+\beta=\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)+\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)+\sin^{-1}\left(\frac{1}{3}\right)+\cos^{-1}\left(\frac{1}{3}\right)$

$$=\frac{\pi}{2}+\frac{\pi}{2}=\pi$$

Also, $\alpha=\frac{\pi}{3}+\sin^{-1}\left(\frac{1}{3}\right)<\frac{\pi}{3}+\sin^{-1}\left(\frac{1}{2}\right)$

As $\sin\theta$ is increasing in $\left[0,\frac{\pi}{2}\right]$.

$$\therefore \quad \alpha<\frac{\pi}{3}+\frac{\pi}{6}=\frac{\pi}{2}$$

Similarly, $\beta>\pi/2$ $\quad\left[\text{since, }\cos\theta\text{ is decreasing in }\left[0,\frac{\pi}{2}\right]\right]$

$\Rightarrow \beta>\frac{\pi}{2}>\alpha \quad\Rightarrow\quad \alpha<\beta$

41. Here, $(AB)(AB)=A(BA)B=A(A^2B)B=A^3B^2$

Now, $(AB)(AB)(AB)=(A^3B^2)AB$

$$=A^3B^2AB=A^3B(BA)B$$

$$=A^3B(A^2B)B=A^3(BA)\cdot AB^2$$

$$=A^3(A^2B)\cdot AB^2$$

$$=A^5BAB^2=A^5\cdot A^2B\cdot B^2=A^7\cdot B^3$$

So, $(AB)^n=A^{2^n-1}\cdot B^n$

$\therefore\ (AB)^{10} = A^{2^{10}-1} \cdot B^{10}$

$\Rightarrow \quad k = 2^{10} - 1 = 1023$

42. Given, functions $f(x) = x^2, x \in R$

and $\quad g(A) = \{x \in R : f(x) \in A\}; A \subseteq R$

Now, for $S = [0, 4]$

$$g(S) = \{x \in R : f(x) \in S = [0, 4]\}$$
$$= \{x \in R : x^2 \in [0, 4]\} = \{x \in R : x \in [-2, 2]\}$$

$\Rightarrow \quad g(S) = [-2, 2]$

So, $f(g(S)) = [0, 4] = S$

Now, $\quad f(S) = \{x^2 : x \in S = [0, 4]\} = [0, 16]$

and $g(f(S)) = \{x \in R : f(x) \in f(S) = [0, 16]\}$

$$= \{x \in R : f(x) \in [0, 16]\}$$
$$= \{x \in R : x^2 \in [0, 16]\}$$
$$= \{x \in R : x \in [-4, 4]\} = [-4, 4]$$

From above, it is clear that $g(f(S)) \neq g(S)$.

43. It is given that $x = 2\sin\theta - \sin 2\theta$

and $y = 2\cos\theta - \cos 2\theta, \theta \in [0, 2\pi]$

$$\therefore \quad \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{-2\sin\theta + 2\sin 2\theta}{2\cos\theta - 2\cos 2\theta} = \frac{\sin 2\theta - \sin\theta}{\cos\theta - \cos 2\theta}$$

$$\therefore \quad \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{d\theta}\left(\frac{dy}{dx}\right) \times \frac{d\theta}{dx}$$

$$= \frac{d}{d\theta}\left(\frac{\sin 2\theta - \sin\theta}{\cos\theta - \cos 2\theta}\right) \times \frac{1}{\frac{dx}{d\theta}}$$

$$= \frac{(\cos\theta - \cos 2\theta)(2\cos 2\theta - \cos\theta) - (\sin 2\theta - \sin\theta)(-\sin\theta + 2\sin 2\theta)}{(\cos\theta - \cos 2\theta)^2} \times \frac{1}{(2\cos\theta - 2\cos 2\theta)}$$

$$\therefore \quad \left.\frac{d^2y}{dx^2}\right|_{\theta=\pi} = \frac{(-1-1)(2+1) - (0-0)(-0+0)}{2(-1-1)^3} = \frac{-2\times 3}{-2\times 8} = \frac{3}{8}$$

44. Given, $\quad \phi(x) = f(x) + f(2-x), \forall\ x \in (0, 2)$

$\Rightarrow \quad \phi'(x) = f'(x) - f'(2-x)$ ...(i)

Also, we have $f''(x) > 0\ \forall\ x \in (0, 2)$

$\Rightarrow \quad f'(x)$ is a strictly increasing function $\forall\ x \in (0, 2)$.

Now, for $\phi(x)$ to be increasing,

$$\phi'(x) \geq 0$$

$\Rightarrow \quad f'(x) - f'(2-x) \geq 0$ [using Eq. (i)]

$\Rightarrow \quad f'(x) \geq f'(2-x) \Rightarrow x > 2-x$ [$\because f'$ is a strictly increasing function]

$\Rightarrow \quad 2x > 2 \Rightarrow x > 1$

Thus, $\phi(x)$ is increasing on (1, 2).

Similarly, for $\phi(x)$ to be decreasing,

$$\phi'(x) \leq 0$$

$\Rightarrow \quad f'(x) - f'(2-x) \leq 0$ [using Eq. (i)]

$\Rightarrow \quad f'(x) \leq f'(2-x) \Rightarrow x < 2-x$ [$\because f'$ is a strictly increasing function]

$\Rightarrow \quad 2x < 2 \Rightarrow x < 1$

Thus, $\phi(x)$ is decreasing on (0, 1).

45. Let E_1 : The event that fair die is chosen .

E_2 : The event that rigged die is chosen.

A : The event that tossing die three times it shows 5 on all the faces.

$$P(E_1) = \frac{3}{4}, P(E_2) = \frac{1}{4}$$

$$P\left(\frac{A}{E_1}\right) = {}^3C_3 \left(\frac{1}{6}\right)^3 = \frac{1}{6^3}, P\left(\frac{A}{E_2}\right) = 1$$

Now, $$P\left(\frac{E_2}{A}\right) = \frac{P(E_2)\,P\left(\frac{A}{E_2}\right)}{P(E_1)\,P\left(\frac{A}{E_1}\right) + P(E_2)\,P\left(\frac{A}{E_2}\right)}$$

$$= \frac{\frac{1}{4} \times 1}{\frac{3}{4} \times \frac{1}{6^3} + \frac{1}{4} \times 1} \qquad = \frac{1}{\frac{3}{216} + 1} = \frac{216}{219}$$

46. Equation of plane passing through the points (2, 1, 0), (4, 1, 1) and (5, 0, 1) is

$$\begin{vmatrix} x-2 & y-1 & z-0 \\ 4-2 & 1-1 & 1-0 \\ 5-2 & 0-1 & 1-0 \end{vmatrix} = 0 \Rightarrow \begin{vmatrix} x-2 & y-1 & z \\ 2 & 0 & 1 \\ 3 & -1 & 1 \end{vmatrix} = 0$$

$\Rightarrow \quad (x-2)(0+1) - (y-1)(2-3) + z(-2-0) = 0$

$\Rightarrow \quad x - 2 + y - 1 - 2z = 0$

$\Rightarrow \quad x + y - 2z - 3 = 0$ ...(i)

Now, let the image of point $R(2, 1, 6)$ w.r.t. plane (i) is (x_1, y_1, z_1), then

$$\frac{x_1-2}{1} = \frac{y_1-1}{1} = \frac{z_1-6}{-2} = -2\left(\frac{2+1-12-3}{1^2+1^2+(-2)^2}\right)$$

$$\Rightarrow \quad \frac{x_1-2}{1} = \frac{y_1-1}{1} = \frac{z_1-6}{-2} = \frac{2 \times 12}{6} = 4$$

$\Rightarrow \quad x_1 = 6, y_1 = 5, z_1 = -2$

$\therefore$ Image of point $R(2, 1, 6)$ w.r.t. plane (i) is (6, 5, –2).

47. Given $y = x^{1/x}$

$$\ln y = \frac{1}{x}\ln x = f(x)$$

$$\therefore \quad f'(x) = \frac{1-\ln x}{x^2}$$

For maxima and minima, $\frac{1-\ln x}{x^2} = 0$

$$\Rightarrow \quad 1-\ln x = 0 \quad \Rightarrow \quad x = e$$

Now, $$f''(x) = \frac{-x-2(1-\ln x)x}{(x^2)^2}$$

At $$x = e$$

$$f''(x) = -ve < 0$$

Hence, at $x = e$

$f(x)$ is maximum and maximum value of $x^{1/x} = e^{1/e}$.

48. $\because$ Equation of normal at $P(1, 1)$ is

$$ay + x = a + 1 \quad \text{[given]}$$

$\because$ Slope of normal at $(1, 1) = -\frac{1}{a}$

$\therefore$ Slope of tangent at $(1, 1) = a$ ...(i)

Also, given $\frac{dy}{dx} \propto y$

$$\Rightarrow \quad \frac{dy}{dx} = ky$$

$$\left[\frac{dy}{dx}\right]_{(1,1)} = k = a \quad \text{[from Eq. (i)]}$$

Then, $\frac{dy}{dx} = ay$

$$\Rightarrow \quad \frac{dy}{y} = a\,dx$$

$$\Rightarrow \quad \ln|y| = ax + C$$

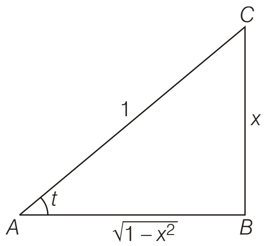
$\because$ It is passing through (1, 1),then

$$c = -a$$

$$\Rightarrow \quad \ln|y| = a(x-1)$$

$$\Rightarrow \quad |y| = e^{a(x-1)}$$

49. Let $I = \int \frac{\sin^{-1} x}{(1-x^2)\sqrt{1-x^2}} dx$



On putting, $\sin^{-1} x = t \Rightarrow \frac{1}{\sqrt{1-x^2}} = \frac{dt}{dx} \Rightarrow dx = \sqrt{1-x^2}\, dt$

$\therefore \quad I = \int \frac{t}{(1-\sin^2 t)\sqrt{1-x^2}} \cdot \sqrt{1-x^2} dt$

$= \int \frac{t}{(1-\sin^2 t)} dt = \int \underset{\text{I}}{t} \underset{\text{II}}{\sec^2} dt$ $\quad [\because 1-\sin^2 t = \cos^2 t]$

$= t\int \sec^2 t dt - \int \left[\frac{d}{dt} t \int \sec^2 t dt\right] dt$ $\quad$ [integration by parts]

$= t \tan t - \int \tan t\, dt = t \tan t + \log \cos t + C$

$= (\sin^{-1} x) \frac{x}{\sqrt{1-x^2}} + \log(\sqrt{1-x^2}) + C$

[from figure $\tan t = \frac{x}{\sqrt{1-x^2}}$ and $\cos t = \sqrt{1-x^2}$]

50. Since, $a_1, a_2, a_3, \ldots, a_n$ are in AP with common difference 5.

$\Rightarrow \quad a_2 - a_1 = a_3 - a_2 = \ldots = a_n - a_{n-1} = 5$

Now, $\quad T_1 = \tan^{-1} \frac{5}{1+a_1 a_2} = \tan^{-1} \frac{a_2 - a_1}{1 + a_2 a_1}$

$= \tan^{-1} a_2 - \tan^{-1} a_1$

Similarly, $\quad T_2 = \tan^{-1} a_3 - \tan^{-1} a_2$

$T_3 = \tan^{-1} a_4 - \tan^{-1} a_3$

$\vdots \quad \vdots \quad \vdots$

$T_{n-1} = \tan^{-1} a_n - \tan^{-1} a_{n-1}$

On adding all, we get

Required sum $= \tan^{-1} a_n - \tan^{-1} a_1 = \tan^{-1} \frac{a_n - a_1}{1 + a_n a_1}$

$= \tan^{-1} \frac{a_1 + 5(n-1) - a_1}{1 + a_n a_1}$

$= \tan^{-1} \frac{5(n-1)}{1 + a_n a_1}$